# Counterfactual Calculation in Multiple Discrete Continuous Choice with Multiple Budget Constraints *

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**Abstract**

I extended the efficient greedy algorithm provided by Pinjari and Bhat (2011) for the single constraint to an iterative method accommodating more than one budget constraint in the multiple discrete-continuous choice model. This can be used to solve the problem where, for instance, there is an extra binding purchase obligation from some currently-hold contracts. Monte Carlo simulation has demonstrated that, for an optimization with extra binding minimum purchase, which seems prohibitively large to solve by traversing all the possible choice sets, this iterative method can obtain the solution with an elapsed time of less than 0.2 seconds.

*All errors are my own.

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## 1 Quick Introduction

Multiple Discrete-Continuous demand system has become increasingly popular recently, especially after Bhat (2005)'s seminal work on MDCEV[1]. Since multiple discrete-continuous choice model does not have an analytical solution, the counterfactual calculation could be computationally hard, or even infeasible. Theoretically, all possible combinations of non-zero set combinations must be checked with the possibility of a corner optimizer. In this note, I extended the greedy algorithm proposed in Pinjari and Bhat (2011) to an iterative one, which can deal with the counterfactual simulation with more than one budget constraint. The greedy algorithm in Pinjari and Bhat (2011) is a close enough approximation to the true solution, and it is always computationally feasible.

To illustrate the practical problem at hand, I target these multiple discrete-continuous models through cost minimization. Similar logic applies to utility maximization directly. Suppose there is one company with **I** subsidiaries that needs to purchase some inputs for each of its subsidiaries. Budget constraints are imposed on each subsidiary, and they must purchase at least that amount of input. One practical example that could justify the model's basic setting comes from the fuel-fired power plants, which are all operated by a large electricity company. The parent company tries to minimize the cost required to generate a fixed amount of electricity.

The cost minimization problem for the company **I**, over all the subsidiaries i, by choosing a subset of goods inside each subsidiary's $J_{it}$ in the spot market, which can be written as,

$$\begin{aligned} \min_{Q_{ijt}, j\in J_{it}} \; TC_{It} &= \sum_{i=1}^{I}\sum_{j=1}^{J_{it}} \frac{1}{\alpha+1} e^{C_{ijt}}[(Q_{ijt}+1)^{1+\alpha}-1] \\ \text{s.t} \quad \sum_{j=1}^{J} Q_{ijt}h_{jt} \geq E_{it}; &\quad \forall\, Q_{ijt}\geq 0,\ \forall\, i\in I,\ j\in J_{it}; \quad C_{ijt} = X_{ijt}\beta + \epsilon_{ijt}. \end{aligned} \tag{1}$$

where $E_{it}$ is the requirement for the inputs to purchase, indicating that at least $E_{it}$ amount of goods need to be consumed. $h_{jt}$ is the characteristics attached to the quantity that enters into the budget constraints. In this example, $E_{it}$ could be the total energy input required to generate electricity, and $h_{jt}$ is the heat content measured in MMBtu per short ton. $\epsilon_{ijt}$ is the error term. Without loss of generality, $h_{jt}$ is assumed not vary with subsidiary i. In Bhat (2005), it takes the form of T1EV. $\epsilon_{ijt}$ is exogenous towards all characteristics in $X_{ijt}$. So that if the price is a part of

1. MDCEV is the multiple discrete-continuous model with the error term follows a Logit distribution, so that there would be an analytical MLE objective function.

$X_{ijt}$, $\epsilon_{ijt}$ should not be realized when prices are determined[2].

Typically, in the multiple discrete-continuous model, a diminishing return to scale parameter $\alpha$ is introduced; otherwise, firm i would always purchase from the lowest cost until $E_{it}$ is satisfied. Or in other words, there would be no cross-substitution in the purchase bundle. To have a valid cost minimization problem, there needs to be $\alpha > 0$. Each subsidiary has its own requirement $E_{it}$, and each $E_{it}$ enters into the system as an equality constraint. So totally, there is **I** budget constraints to satisfy for optimization. I assume that $E_{it}$ is exogenous or predetermined before plants come to the stage to determine the varieties to purchase from.

Then write the Lagrange function of (1) with

$$\mathscr{L} = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{1}{\alpha+1}e^{C_{ijt}}[(Q_{ijt}+1)^{1+\alpha}-1] + \sum_{i=1}^{I}\lambda_i(E_{it} - \sum_{j=1}^{J}Q_{ijt}h_{jt}) \tag{2}$$

The coefficients can be estimated using the MLE derived in Bhat (2005) and Bhat (2008). The counterfactual calculation for this model involves calculating the optimal discrete-continuous choices $Q_{ijt}$ from the known parameters $\alpha$, $\beta$, and a realization of $\boldsymbol{\epsilon}$. Or the expectation $E[Q_{ijt}|\boldsymbol{X_{it}}]$ over $\boldsymbol{\epsilon}$ using a large set of draws from the known distribution of $\boldsymbol{\epsilon}$. Since $\epsilon_{ijt}$ is assumed exogenous regarding price, if the counterfactual is regarding price changes, $E[Q_{ijt}|\boldsymbol{X_{it}}]$ would be a better indicator than $Q_{ijt}$ respect to a specific realization of $\epsilon_{it}$.

Unfortunately, unlike the pure discrete or continuous choice problems, the counterfactual optimal $Q_{ijt}$ does not have an analytic solution due to the possibility of corner solutions. Retrieving it is almost equivalent to solving a high-dimensional mixed-integer problem, which could be computationally cumbersome.

To solve this problem, Train and Wilson (2011) decomposes the optimization problem into many one-unit discrete choices and solves it unit-by-unit. Train and Wilson (2011) claims that their method can alleviate the computational issue even for high dimensional Pinjari and Bhat (2011), instead, proposes a simple greedy algorithm that provides a good enough approximation to the true solution. Use Pinjari and Bhat (2011), the original problem in (1) is directly solvable with its equivalence to optimizing over all the i in **I** separately, given a set of realized error draws $\boldsymbol{\epsilon}$.

2. If, however, there is price endogeneity, the control function provides a valid correction whenever the separability condition is assumed (Blundell and Matzkin 2014). This does not change the framework since the control variable enters as an extra regressor in $X_{ijt}$.

However, besides the purchase available in the spot market, suppose power generating company **I** currently holds a long-term contract that ties up some of the purchase quotes. At least some coal purchases need to be fulfilled by these contracts. The long-run contract is a common practice in the field of input purchase. Without loss of generality, I assume the long-term contract is on the good J. Usually, a minimum purchase obligation accompanies a typical contract. A maximum quantity preventing buyers from utilizing the price gap too much is also possible. Jha (2019) points out that while minimum purchase obligation is a common practice, ceilings of quantity is a rare phenomenon occuring in around 10% of observed contracts.

If these min and max requirements are stringent and in equilibrium, buyers will not deviate. Then the regular optimization described above may not be able to generate quantities on J that satisfy the obligations such that

$$minQ_{Jt} \leq \sum_{i=1}^{I} Q_{iJt} \leq maxQ_{Jt} \tag{3}$$

into the original minimization (1).

One can proceed by directly solving the constrained nonlinear optimization procedure without using any other information, as in Phaneuf and Herriges (2000). All possible combinations of discrete choices need to be checked for corner solutions. However, for $J$ alternatives of company **I**, roughly $2^J$ possible choice sets to evaluate. This exponentially-increasing set constitutes the main issue hindering the exact solution of (1).

Here, I proceed by exploiting the specific structure of this optimization problem and transform the complicated problem of solving the KKT conditions into a nonlinear optimization that can be solved by single-variable bisection. The organization of this note is as follows. In sections 2 and 3, I developed the algorithm for the case where there is only one long-term contract, or it is suitable to add all the long-run contracts into a single index. I carry on a Monte Carlo simulation in section 4 to demonstrate how fast this iterative greedy algorithm is, compared to those that try to traverse all possible combinations. Section 5 contains a brief discussion about extending this into multiple constraints. And section 6 gives a short conclusion.

## 2 One extra long-term contract with outside option J

Without loss of generality, I would focus on the case where the unconstrained optimization respect to this extra constraint generates purchase quantities $\sum_{i=1}^{I} Q_{iJt} < minQ_{Jt}$. Otherwise (3) can be directly discarded. In this case, the constrained optimization must have $\sum_{i=1}^{I} Q_{iJt} = minQ_{Jt}$ – the minimum quantity obligation should be satisfied exactly. The marginal cost of purchasing good J is monotonically increasing in (1) since $\alpha > 1$. Thus, having more quantity than necessary would increase the total cost. Similar argument holds for the case where $\sum_{i=1}^{I} Q_{iJt} > maxQ_{Jt}$. Hence, with one contract, this optimization problem could be transferred to multiple discrete-continuous problems with extra binding constraints.

Furthermore, I assume that this long-run contract is the outside good, which does not suffer from diminishing returns to scale. Allowing for such an outside good would simplify the model construction in the case like Bhat (2018). Also, it is a reasonable assumption for a composite good. An optimal solution over $Q_{ijt}$ occurs when the marginal cost of inside goods increases until it hits the constant level of the outside option. However, it would generate an unrealistic substitution pattern by eliminating the cross-substitution among inside goods. When good j's price or other characteristics changes, i would substitute the outside option only.

With this extra restriction, (1) becomes,

$$\begin{aligned}
&\min_{Q_{ijt}, j \in J_{it}} TC_{It} = \sum_{i=1}^{I} \sum_{j=1}^{J_{it}} \frac{1}{\alpha+1} e^{C_{ijt}} [(Q_{ijt}+1)^{1+\alpha} - 1] \\
&\text{s.t} \quad \sum_{j=1}^{J} Q_{ijt} h_{jt} \geq E_{it}, \quad \forall\, Q_{ijt} \geq 0,\ \forall\, i \in I,\ j \in J_{it}, \quad C_{ijt} = X_{ijt}\beta + \epsilon_{ijt}. \\
&\qquad \sum_{i=1}^{I} Q_{iJt} = minQ_{Jt}.
\end{aligned} \tag{4}$$

All else is the same, yet now we have an extra minimum obligation constraint.

To solve the optimization problem over $Q_{ijt}$, the Lagrange function that distinguish between the inside and an outside option with a constant marginal cost takes the form as

$$
\begin{aligned}
\mathcal{L} = & \overbrace{\sum_{i=1}^{I}\sum_{j=1}^{J-1}\frac{1}{\alpha+1}e^{C_{ijt}}[(Q_{ijt}+1)^{1+\alpha}-1]}^{\text{Consumption of inside goods}} + \underbrace{\sum_{i=1}^{I} e^{C_{iJt}}Q_{iJt}}_{\text{Consumption of outside option}} \\
& + \overbrace{\sum_{i=1}^{I}\lambda_i(E_{it}-\sum_{j=1}^{J}Q_{ijt}h_{jt})}^{\text{Energy requirement}} + \underbrace{\lambda_{I+1}(minQ_{Jt}-\sum_{i=1}^{I}Q_{iJt})}_{\text{Minimum purchase obligation}}
\end{aligned}
\tag{5}
$$

The KKT conditions take the form,

$$
\begin{aligned}
& e^{C_{ijt}}(Q_{ijt}+1)^{\alpha} = \lambda_i h_{jt}, && \text{if } Q_{ijt} > 0,\ and \quad j \neq J \\
& e^{C_{ijt}} > \lambda_i h_{jt}, && \text{if } Q_{ijt} = 0,\ and \quad j \neq J \\
& e^{C_{iJt}} = \lambda_i h_{Jt} + \lambda_{I+1}, && \text{if } Q_{iJt} > 0 \\
& e^{C_{iJt}} > \lambda_i h_{Jt} + \lambda_{I+1}, && \text{if } Q_{iJt} = 0 \\
& \sum_{j=1}^{J} Q_{ijt}h_{jt} = E_{it}, && \forall i \in I \\
& \sum_{i=1}^{I} Q_{ijt} = minQ_{Jt}
\end{aligned}
\tag{6}
$$

The intuition behind the KKT in (6) is simple. All choice variables $Q_{ijt}$ inside a subsidiary end up with the same $h_{ijt}$-normalized marginal cost. The contract good J, whose purchased quantity is obligated to be higher than the optimal, would generate a higher marginal cost. This leads to $\lambda_{I+1} > 0$ for all the subsidiaries that purchase it. Because there is no diminishing return to scale for good J, the part with $\alpha$ does not enter the condition for outside good J.

If the number of subsidiaries is larger than 1, the system may become hard to solve. A possible way is to optimize over $\lambda_{I+1}$. With current iteration of $\lambda_{I+1}$ as a known variable, for the subsidiaries that purchase J in the original problem, there is $\lambda_i = (e^{C_{iJt}} - \lambda_{I+1})/h_{Jt}$. With the estimated $\lambda_i$, one can directly solve the quantity purchased $Q_{ijt}$. Then one could go back to check whether or not current $\lambda_{I+1}$ is reasonable for subsidiary i. For example, if $\sum_{j=1}^{J-1} Q_{ijt} > E_{it}$, then $\lambda_{I+1}$ is too small for the optimal solution.

The optimization problem can be solved through this $\lambda_{I+1}$ directly. Re-parameterize $\lambda_{I+1}$ as exponential to keep it positive. Following the thread of logic above, I can construct an objective function to minimize over $\lambda_{I+1}$ is

$$min_{\lambda_{I+1}}\Big[\sum_{i=1}^{I}E_{it}-\sum_{i=1}^{I}\sum_{j=1}^{J-1}Q_{ijt}(\lambda_{I+1})\times h_{jt}\Big]^2 \tag{7}$$

The objective function minimizes the distance between the calculated $Q_{ijt}$ and the required $E_i$, which exists only as a function of $\lambda_{I+1}$. Given the current iteration of $\lambda_{I+1}$, all the calculations can be pinned down directly by the intermediate steps. Then with the current $\lambda_{I+1}$, the optimization problem for each subsidiary can be unraveled following the efficient method in Pinjari and Bhat (2011), which automatically allows the corner solutions.

To start this iterative algorithm, the subsidiaries purchasing J in the constrained optimization solution need to be specified. Fortunately, if subsidiary i purchase good J in the original problem (1), it must also purchase in the constrained version (6). Because if not – J purchased in the original problem, but not in the constrained problem – then the extra quantity requirement needs to be covered by $j \neq J$, which brings up the $\lambda_i$ from the original problem (1). Then one cannot have $e^{C_{iJt}} > \lambda_i h_{Jt} + \lambda_{I+1}$ with a non-negative $\lambda_{I+1}$, since in the original case, with positive purchase of J, there is $e^{C_{iJt}} = \lambda_i h_{Jt}$. Thus, with a contradiction inside the necessary KKT conditions in (6), it cannot be an optimal solution.

Another way to form the problem is to optimize over $Q_{1Jt}$ instead, where 1 denotes the subsidiary that starts the iteration. An advantage is, $Q_{1Jt}$ is restricted inside a closed interval, so that a bisection algorithm can be adopted. I will show the details of this in section 3 when there is no outside option.

## 3 One extra long-run contract, and no outside option

Section 2 discusses the case where there is an outside option J, yet such a J may not always exist. Having an outside option is not that reasonable in specific settings. It may be because no common outside option is available, or a particular good without diminishing return to scale is hard to justify. Then the system becomes slightly more complicated since all purchased quantities $Q_{ijt}$ would enter as part of the KKT condition.

In this case, with a binding minimum quantity, the Lagrange function becomes

$$\mathcal{L} = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{1}{\alpha+1}e^{C_{ijt}}[(Q_{ijt}+1)^{1+\alpha}-1] + \sum_{i=1}^{I}\lambda_i(E_i - \sum_{j=1}^{J}Q_{ijt}h_{jt}) + \lambda_{I+1}(minQ_{Jt} - \sum_{i=1}^{I}Q_{iJt}) \tag{8}$$

Then the KKT condition takes the form as

$$e^{C_{ijt}}(Q_{ijt}+1)^{\alpha} = \lambda_i h_{jt}, \qquad \text{if } Q_{ijt} > 0,\ and \quad j \neq J \tag{9}$$

$$e^{C_{ijt}} > \lambda_i h_{jt}, \qquad \text{if } Q_{ijt} = 0,\ and \quad j \neq J \tag{10}$$

$$e^{C_{iJt}}(Q_{iJt}+1)^{\alpha} = \lambda_i h_{Jt} + \lambda_{I+1}, \qquad \text{if } Q_{iJt} > 0 \tag{11}$$

$$e^{C_{iJt}} > \lambda_i h_{Jt} + \lambda_{I+1}, \qquad \text{if } Q_{iJt} = 0 \tag{12}$$

$$\sum_{j=1}^{J} Q_{ijt}h_{jt} = E_{it}, \qquad \forall i \in I \tag{13}$$

$$\sum_{i=1}^{I} Q_{ijt} = minQ_{Jt} \tag{14}$$

The system of equations could be transformed into a lower dimension iterative algorithm, and can be solved by bisection over the quantity purchased over the manually-chosen baseline good[3].

Below are the algorithm I propose in order to compute a new optimal solution of $Q_{ijt}$, given model parameters.

**Initialization:** Solve the original optimization problem (1),

$$\mathcal{L} = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{1}{\alpha+1}e^{C_{ijt}}[(Q_{ijt}+1)^{1+\alpha}-1] + \sum_{i=1}^{I}\lambda_i(E_{it} - \sum_{j=1}^{J}Q_{ijt}h_{jt}) \tag{15}$$

by the greedy algorithm proposed in Pinjari and Bhat (2011). This gives the original calculation of $\widetilde{Q}_{ijt}$. If the set of $\widetilde{Q}_{ijt}$ satisfy the condition in (3), then $\widetilde{Q}_{ijt}$ is an inner solution and the algorithm terminates.

Algorithm 1 below briefly summarizes this algorithm.

**Algorithm 1.** *Approximate an optimal solution to the problem (4),*

1. *Choose a subsidiary that always purchase J and at least one inside good in the optimal solution.*

3. Even without a common outside option, for **I**, a baseline good still needs to be specified as an anchor for other goods to enter as comparison.

*Denote it as firm 1.*

2. *In k-th iteration, given the current upper and lower bound* $[\tilde{Q}^k_{1Jt}, \hat{Q}^k_{1Jt}]$*, set* $Q^k_{1Jt} = (\tilde{Q}_{1Jt} + \hat{Q}_{1Jt})/2$*. Calculate following KKT in (9), (10) and (11) to get* $\lambda_1$ *and* $\lambda_{I+1}$*.*

3. *With* $\lambda_{I+1}$*, from the KKT in (9), (10) and (11) also, can obtain the rest of quantity purchases* $Q_{i'jt}$*, for all other subsidiaries,* $i' \in \boldsymbol{I}$*,* $i' \neq 1$*.*

4. *Update the upper and lower bound* $[\tilde{Q}^{k+1}_{1Jt}, \hat{Q}^{k+1}_{1Jt}]$*, according to the violation of minimum purchase obligation.*

5. *Set* $k = k + 1$ *and iterate until some termination condition satisfied.*

Then detailed explanations and discussions about how to proceed with each step is illustrated in the paragraphs below.

**Step 1, obtain the baseline subsidiary.** This algorithm must start at a subsidiary that purchases from inside choices and long-term obligations at the optimal solution. The requirement of inside good purchases is a must, otherwise $\lambda_{I+1}$ cannot be pinned down by (9) and (11).

There will always be such a subsidiary if the minimum long-term obligation is not too large such that $minQ_{Jt} \times h_{Jt}$ equals $\sum_{i \in \boldsymbol{I}} E_{it}$. In order to find it to begin with, firstly, I figure out the subsidiaries that cannot have all their $E_{it}$ purchased from J from the original (1), by assuming every subsidiary fulfilling the $E_{it}$ by J and finding the appropriate cutoff.

For $i \in I$, suppose it is optimal for i to purchase all its energy requirement from J, then this would generate a lower bound for $\lambda_{I+1}$ from the KKT condition. If all $E_{it}$ are purchased from J, then all other goods inside the choice set would not be purchased. For the KKT condition in (10), there would be $e^{C_{ijt}} > \lambda_i h_{jt}$, for all $j \neq J$. Under current draw of $\boldsymbol{\epsilon}_i$, and coefficients estimated, I can find that, in an optimal solution, the largest $\lambda_i$ among $i \in \boldsymbol{I}$ is $\bar{\lambda}_i = \min_{j \neq J} e^{C_{ijt}}/h_{jt}$. And by the KKT condition in (11), $e^{C_{iJt}}(Q_{iJt}+1)^{\alpha} = \lambda_i h_{Jt} + \lambda_{I+1}$, holding the left hand side constant, the minimum $\lambda_{I+1}$ that is consistent with this optimization system is $\underline{\lambda}_{I+1} = e^{C_{iJt}}(E_{it}/h_{Jt}+1)^{\alpha} - \lambda_i h_{Jt}$. To put it in another way, the logic behind can be summarized as,

$$e^{C_{ijt}} > \lambda_i h_{jt} \Rightarrow \bar{\lambda}_i = \max\ \lambda_i = \min \left[\frac{e^{C_{ijt}}}{h_{jt}}\right] \Rightarrow \underline{\lambda}_{I+1} = \min\ \lambda_{I+1} = e^{C_{iJt}} \left[\frac{E_{it}}{h_{Jt}} + 1\right]^{\alpha} - \bar{\lambda}_i h_{Jt} \quad (16)$$

However, this min $\underline{\lambda}_{I+1}$ may already be "too large", mainly because $\lambda_{I+1}$ is shared across all i in

**I**. This $\lambda_{I+1}$ could lead to total purchase summarized across all $i \in I$ larger than the $minQ_{Jt}$. To check it, given the $\underline{\lambda}_{I+1}$, I solve all $i' \in I, i' \neq i$. If it is indeed too large, I would have $\sum_{i=1}^{I} Q_{iJt} > minQ_{Jt}$, which is contradicted with the binding constraint in optimization.

Then I pick the first subsidiary that satisfies the two conditions – (i) causes the $\underline{\lambda}_{I+1}$ "too large" from (16) and (ii) purchase some J in the original problem (1), as the baseline subsidiary, and denote it as 1. (i) guarantees that in the optimal solution, subsidiary 1 would purchase from inside goods, while (ii) guarantees that subsidiary 1 would purchase from J. If (i) is never "too large", then the optimal solution would be everyone purchasing from J only. If no good J is purchased by any subsidiary in original (1), which is possible but rare, then start with the subsidiary that encounters the lowest extra cost when switching from the inside goods to J until the minimum purchase obligation is fulfilled entirely. The subsidiary lies in the cut-off point must purchase good J in the (4).

**Step 2: solve the $\lambda_{I+1}^{k}$.** An optimal solution with the extra minimum quantity obligation must have $Q_{1Jt}$ lies between the unconstrained result $\widetilde{Q}_{1Jt}$, and smaller than or equal to $\hat{Q}_{1Jt} = min\{E_{1t}/h_{Jt}, minQ_{Jt}\}$. By this I have a closed interval $[\widetilde{Q}_{1Jt}, \hat{Q}_{1Jt}]$ for possible $Q_{1Jt}$. The initial bisection generates

$$Q_{1Jt}^{k} = \frac{\widetilde{Q}_{1Jt} + \hat{Q}_{1Jt}}{2} \tag{17}$$

Then with a known $Q_{1Jt}^{k}$ at current k-th iteration, 1's optimization problem can be solved, using Pinjari and Bhat (2011). It is just as if, solve the original problem for 1, with $E'_{1t} = E_{1t} - Q_{1Jt}h_{Jt}$. Then current iteration of $\lambda_{I+1}^{k}$ are obtained by (11).

**Step 3: solve the purchased quantities for the rest of the model.** Let $A_J$ denotes the set of subsidiaries that purchase a positive quantity of J in the original solution to (1). With a known $\lambda_{I+1}^{k}$, their optimization problem can be solved through another layer of bisection, though an analytic solution as in Pinjari and Bhat (2011) is no longer available with a $\lambda_{I+1}^{k} > 0$.

To see this, for a subsidiary $i \in A_J$, the $\lambda_i$ is solved by the budget constraint, and with $\lambda_{I+1} = 0$, the budget constraint (13) combined with (11) becomes a separable function of $\lambda_i$ only,

$$\sum_{j=1}^{J} \left[ \left( \frac{\lambda_i h_{jt}}{e^{C_{ijt}}} \right)^{\frac{1}{\alpha}} - 1 \right] h_{jt} = \lambda_i^{\frac{1}{\alpha}} \sum_{j=1}^{J} \left[ \left( \frac{h_{jt}}{e^{C_{ijt}}} \right)^{\frac{1}{\alpha}} - 1 \right] h_{jt} = E_{it} \tag{18}$$

Hence $\lambda_i$ can be separated and solved analytically. However, with $\lambda_{I+1} > 0$, the budget constraint then becomes

$$\sum_{j=1}^{J-1}\left[\left(\frac{\lambda_i h_{jt}}{e^{C_{ijt}}}\right)^{\frac{1}{\alpha}} - 1\right]h_{jt} + \left(\frac{\lambda_i h_{jt} + \lambda_{I+1}}{e^{C_{iJt}}}\right)^{\frac{1}{\alpha}} - h_{Jt} = E_{it} \tag{19}$$

It is no longer separable, and cannot be solved easily in an analytical way as in the separable case. So another layer of bisection over $Q_{iJt}$ is needed to pin down the $\lambda_i, i \neq 1$.

For the bisection method inside a subsidiary i, I starts with a $Q_{iJt}$ at the middle point of the interval $[\widetilde{Q}_{iJt}, \hat{Q}_{iJt}]$. $\widetilde{Q}_{iJt} = 0$, $\hat{Q}_{iJt} = E_{it}/h_{Jt}$. Given $\lambda_{I+1}$ and $Q_{iJt}$, $\lambda_i$ can be solved instantly from the KKT conditions (9) and (11), so would the $Q_{ijt}, j \neq J$ instantaneously. The end points $\widetilde{Q}_{iJt}$, $\hat{Q}_{iJt}$ in the bisection is updated as

$$\begin{aligned} \widetilde{Q}_{iJt} &= Q_{iJt}, \quad \text{if } E_{it} - \sum_{j=1}^{J} Q_{ijt}h_{jt} > 0 \\ \hat{Q}_{iJt} &= Q_{iJt}, \quad \text{if } E_{it} - \sum_{j=1}^{J} Q_{ijt}h_{jt} < 0 \end{aligned} \tag{20}$$

The update in (20) is intuitive. If i purchase too little $\sum_{j=1}^{J} Q_{ijt}h_{jt}$, compared to the total requirement $E_i$, then it means the $\lambda_i$ is too small. Given the $\lambda_{I+1}$ hold constant, $Q_{iJt}$ is too small – so needs the potential solution of $Q_{iJt}$ to lie in a larger interval. Converse logic applies when ends in the second equation of (20). And the bisection stops when the interval becomes smaller than the threshold $\delta_i$, $|\widetilde{Q}_{iJt} - \hat{Q}_{iJt}| < \delta_i$. $\delta_i$ is a small positive number.

Then repeatedly solve the bisection on $Q_{i'Jt}$ for all i' $\in A_J$. Also note that there is some possibility such that a subsidiary that does not purchase from contract in (1), would purchase under the minimum obligation. To account for this, for the subsidiary i with original $\widetilde{Q}_{iJt} = 0$, firstly, I assume the choice set still contains inside goods only, and solve it follows Pinjari and Bhat (2011). With $\lambda_i$, $\lambda_{I+1}$ at hand, instantaneously one can check whether there is

$$e^{C_{i'Jt}} > \lambda_{i't}h_{Jt} + \lambda_{I+1}$$

If not, then add this $i'$ as part of the $A_J$.

**Step 4: update $Q_{1Jt}^{k}$ to $Q_{1Jt}^{k+1}$.** For the last power plant $i'$ that would purchase a positive quantity of J, due to the binding $\sum_{i=1}^{I} Q_{iJt} = minQ_{Jt}$, $Q_{i'Jt}$ is obtained directly. The objective function for the outer bisection over $Q_{1Jt}$ is the distance between the obtained total energy content and the required $E_{i'}$, as in (7). Built on this, the outer bisection on $[\widetilde{Q}_{1Jt}, \hat{Q}_{1Jt}]$ is updated by,

$$\begin{aligned} \widetilde{Q}_{1Jt} &= Q_{1Jt}, \quad \text{if } E_{i't} - \sum_{j=1}^{J} Q_{i'jt} h_{jt} > 0 \\ \hat{Q}_{1Jt} &= Q_{1Jt}, \quad \text{if } E_{i't} - \sum_{j=1}^{J} Q_{i'jt} h_{jt} < 0 \end{aligned} \tag{21}$$

**Step 5:** Repeat the entire iteration until, for a small positive number $\delta$, $|\widetilde{Q}_{1Jt} - \hat{Q}_{1Jt}| < \delta$.

Therefore, the multiple discrete-continuous models with an extra budget constraint could be transferred into a nonlinear programming problem with one choice parameter $Q_{1Jt}$ lying inside a closed interval. A rough justification that the KKT conditions for (8) could be approximated by this iterative greedy bisection algorithm is sketched below.

For $Q_{1Jt}$ smaller than the optimal level, for subsidiary 1, more quantity would be purchased from the inside goods, then $\lambda_1$ would be larger, and $\lambda_{I+1}$ would be smaller than its optimal level. Correspondingly, the rest subsidiaries would purchase too less good J since their purchase of good J is constrained by $\lambda_{I+1}$). Thus, it leaves good J with more than necessary quantity for the subsidiary $i'$ that enters at last. Then $i'$ would generate a higher total purchase than $E_{i't}$, if both the KKT condition for the inside goods and J are satisfied. With the KKT condition (11), $e^{C_{i'Jt}}(Q_{i'Jt} + 1)^{\alpha} = \lambda_{i'} h_{Jt} + \lambda_{I+1}$ forced to be satisfied, if $Q_{i'Jt}$ is larger than the optimal, then holding $\lambda_{I+1}$ constant, $\lambda_{i'}$ would be larger, which leads to unnecessarily more purchase of good 1 to good J-1 for the last i'.

To correct for this, $Q_{1Jt}$ needs to increased. Because of the monotonically increasing marginal cost for all power plants, I would move along a fixed direction. If $Q_{1Jt}$ increases, so would $\lambda_{I+1}$; with larger $\lambda_{I+1}$, all power plants in between would purchase more J, and leave less J for $i'$. The left hand side of (11), $e^{C_{i'Jt}}(Q_{i'Jt} + 1)^{\alpha} = \lambda_{i'} h_{Jt} + \lambda_{I+1}$, for $i'$, decreases, while $\lambda_{I+1}$ increases, so that $\lambda_{i'}$ decreases, and it becomes less likely to have extra purchase as before.

On the contrary, for a larger $Q_{1Jt}$, less quantity would be purchased from the inside goods by subsidiary 1, then $\lambda_1$ is smaller, and $\lambda_{I+1}$ would be larger. Correspondingly, the rest subsidiaries would purchase too much good J and leave the $i'$ with a relatively smaller $\lambda_{i'}$. Hence, either would it lead to negative long-run quantity purchased or purchase less than the requirement $E_{i'}$. In this case, slightly increasing $Q_{1Jt}$ would help.

In summary, Algorithm 1 uses the interconnection of the KKT conditions implied by the constrained optimization, to transform the a high-dimensional mixed-integer problem into a nonlinear bisection over one variable $Q_{1Jt}$. Inside the algorithm nests Pinjari and Bhat (2011). As for the case where the maximum is violated when solving (1), $\sum_{i=1}^{I} Q_{iJt} > maxQ_{Jt}$, the similar logic applies, but under the condition such that the inside goods purchased in (1), must also be purchased in the constrained case. So for the binding maximum quantity, I can start with a subsidiary i that purchases from inside goods, and then follow the similar path above.

## 4 Monte Carlo Simulation

Then I conduct Monte Carlo simulation regarding the performance of this algorithm, with settings discussed below.

Suppose the company **I** operates three subsidiaries. $\epsilon$ would be independent across both goods and subsidiaries. I generate many sets of draws from the known distribution of $\epsilon$, and I focus on one particular realization of $\epsilon$. The company **I** holds a long-term contract on good J, which suffers from diminishing return to scale still, with the original outcome from (1) not satisfying the minimum purchase quantity obligation.

Then I generate a bunch of tests by varying a bunch of model features, including, (i) number of inside goods; (ii) number of inside goods that purchased by each subsidiary from solving (1); (iii) number of subsidiaries purchase the long-term J from solving (1); (iv) the quantity of long-term good J purchased from solving (1); (v) the minimum purchase obligations, which is picked randomly; (vi) the changes in inside good purchase from (1) to (4). For instance, in (1), all inside goods are purchased; but in (4), some inside goods may no longer be purchased.

Those model features are governed by changing the linear product characteristics entering (1). That is, I come up with a set of product characteristics so that the number of good purchased in the original problem is 2 for every subsidiary. By changing these features, I come to different data generating process. Other parameters include diminishing return to scale $\alpha = 1.5$, and parameters for the baseline cost in (1), $\beta = [0.2, 0.5, 0.04, -0.25, 0.1]$. I only do the counterfactual part, so I assume that I know all the parameters. Moreover, this counterfactual is specific to a realization of

$\boldsymbol{\epsilon}$, drawn from standard T1EV. If the expectation is needed instead, one must repeat this procedure many times, which emphasizes the importance of a computational-feasible approximation.

To compare with the iterative greedy algorithm, I refer to the brutal-force fmincon solver over all the possibilities on which goods to purchase. The fmincon in Matlab only searches for an interior solution, so I need to review all the candidate combinations of discrete choices. I did not simply examine all the $2^{I\times J}$ candidates. Extra conditions provide some information to rule out the candidates that would not be an optimal solution. For example, the subsidiaries purchasing from the contract in the original case must also purchase in the constrained problem. In this simple case, I would usually not expect two subsidiaries to purchase from contracts only, and so on. I can somehow exclude some combinations by these before traversing all possible cases. Besides, to speed up and overcome the convergence problem that fmincon would have, I start from a feasible initial value.

Note that, I do not know the true $Q_{ijt}$ even in the simulation. I consider the results to be reasonably correct if both algorithms reach the exact the same optimal $Q$.

Table 1 shows the simulation results in different model settings. First, note that the elapsed times are growing very fast for fmincon. With five inside goods, the computational time is already 578 seconds, more than 5000 times of the iterative method. For the number of inside goods growth to 10, a relatively small number of goods in many real applications, the fmincon becomes infeasible to traverse all purchase combinations. At the same time, the iterative greedy algorithm greatly improves the computation speed when both methods give back the same optimization results. The iterative algorithm makes the problem not only solvable but also computationally very efficient. The elapsed time for iterative greedy does not change much as the number of inside goods expands to 10. Less than 0.2 seconds of elapsed time is almost negligible in most cases.

Note that the elapsed time is for one set of randomly picked $\boldsymbol{\epsilon}$. If $E[Q_{ijt}|\boldsymbol{X}]$ rather than the $\boldsymbol{\epsilon}$-connected $Q_{ijt}$ is needed, the computational improvement for a single $\boldsymbol{\epsilon}$ would be magnified by the number of $\boldsymbol{\epsilon}$ drawn to approximate the integration numerically. And whenever the counterfactual involves a structural supply side, the Nash-Bertrand competition, for instance, the expected quantities are required to be calculated repeatedly. Obviously, in those cases, the dominance of the iterative algorithm with respect to the computationally expensive or infeasible fmincon-based would be more prominent.

## 5 Extension to more than one extra constraint

Here I provide a short description about how this might extend into the case where there are more than one extra constraints. Suppose company **I** currently hold two long-term contracts, both with binding minimum purchase obligation. The problem comes in at the Step 2 in Algorithm 1, where the baseline subsidiary that would purchase from contracts in an optimal solution needs to be pinned down.

In this case, one possible solution would be traversing over all $i \in \boldsymbol{I}$. Treating each $i \in \boldsymbol{I}$ as the baseline subsidiary and solve the system as Algorithm 1 indicates. Then compare the optimal value implied in each case. Usually the number of subsidiaries inside a company $\boldsymbol{I}$ is much less than all the possible combinations of discrete choices – making traversing less computationally costly.

Of course, more is involved when there are more than one extra constraints. For instance, the optimal may happens at the place where there is no $i \in \boldsymbol{I}$ that purchase from both contracts and an inside spot good. For a rigorous extension towards this more complicated setting in multiple discrete-continuous choice, further analysis is needed.

## 6 Conclusion

These computational notes provide an iterative greedy algorithm to solve for the counterfactual of the multiple-discrete continuous model with a complicated structure in budget constraints. I demonstrate the algorithm using a hypothetical cost minimization problem between a power generating company and coal producers with one extra minimum purchase obligation from a currently-hold long-term contract. The simulation shows that the iterative method makes the problem feasible and also provides massive computational gains.

# References


Bhat, Chandra R. 2005. "A Multiple Discrete-Continuous Extreme Value Model: Formulation and Application to Discretionary Time-Use Decisions." *Transportation Research Part B: Methodological* 39 (8): 679–707.

———. 2008. "The Multiple Discrete-Continuous Extreme Value (MDCEV) Model: Role of Utility Function Parameters, Identification Considerations, and Model Extensions." *Transportation Research Part B: Methodological* 42 (3): 274–303.

———. 2018. "A New Flexible Multiple Discrete-Continuous Extreme Value (MDCEV) Choice Model." *Transportation Research Part B: Methodological* 110:261–279.

Blundell, Richard, and Rosa L. Matzkin. 2014. "Control functions in Nonseparable Simultaneous Equations Models." *Quantitative Economics* 5 (2): 271–295.

Jha, Akshaya. 2019. "Dynamic Regulatory Distortions: Coal Procurement at U.S. Power Plants." Working Paper. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3330740.

Phaneuf, Daniel J., and Joseph A. Herriges. 2000. "Choice set definition issues in a Kuhn-Tucker model of recreation demand." *Marine Resource Economics* 14 (4): 343–355.

Pinjari, Abdul, and Chandra R. Bhat. 2011. "Computationally Effcient Forecasting Procedures for Kuhn-Tucker Consumer Demand Model Systems: Application to Residential Energy Consumption Analysis." Technical paper, Department of Civil  Environmental Engineering, University of South Florida. July.

Train, Kenneth, and Wesley W. Wilson. 2011. "Coal Demand and Transportation in the Ohio River Basin: Estimation of a Continuous/Discrete Demand System with Numerous Alternatives." Working Paper, https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.491.4959&rep=rep1&type=pdf.

Table 1: A simple simulation on the performance of this iterative greedy algorithm.

| | Test 1 | Test 2 | Test 3 | Test 4 | Test 5 | Test 6 |
|---|---|---|---|---|---|---|
| **Settings** | | | | | | |
| (i) # inside goods | 3 | 3 | 3 | 3 | 5 | 10 |
| (ii) # of inside goods purchased in (1) | 3 | 3 | 3 | 2 | 5 | 10 |
| (iii) # of plants purchase J in (1) | 3 | 3 | 2 | 3 | 3 | 3 |
| (iv) $\sum_{i\in \boldsymbol{I}} Q_{iJt}$ from solving (1) | 165.961 | 165.961 | 98.380 | 180.026 | 105.635 | 56.017 |
| (v) Minimum Obligation | 171 | 310 | 108 | 185 | 111 | 62 |
| (vi) Changes in purchase set from (1) to (4) | No | Yes | No | No | No | No |
| **Performance** | | | | | | |
| Elapsed time (s) **fmincon** | 5.890 | 8.992 | 10.890 | 0.535 | 578.721 | Infeasible |
| Elapsed time (s) **Iterative Greedy** | 0.082 | 0.074 | 0.084 | 0.073 | 0.110 | 0.144 |